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Three Topological Problems about Integral Functionals on Sobolev Spaces

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Abstract. In this paper, I propose some problems, of topological nature, on the energy functional associated to the Dirichlet problem

$$\begin{cases} -\Delta u = f(x,u) \text{ in } \Omega \\ u_{|\partial\Omega} = 0 \,. \end{cases} \qquad (P_f)$$

Positive answers to these problems would produce innovative multiplicity results on problem (P_f).

Key words: Disconnectedness, Energy functional, Isolated point, Local minimum

In the present very short paper, I wish to propose some problems, of topological nature, on the energy functional associated to the Dirichlet problem

$$\begin{cases} -\Delta u = f(x,u) \text{ in } \Omega \\ u_{|\partial\Omega} = 0 \,. \end{cases} \qquad (P_f)$$

and explain their motivations as well.

So, let $\Omega \subset \mathbf{R}^n$ $(n \geqslant 3)$ be an open bounded set. Put $X = W_0^{1,2}(\Omega)$. Consider X with the usual norm $\|u\| = (\int_\Omega |\nabla u(x)|^2 \, dx)^{\frac{1}{2}}$. For $q > 0$, denote by $\mathcal{A}_q$ the class of all Carathéodory functions $f : \Omega \times \mathbf{R} \to \mathbf{R}$ such that

$$\sup_{(x,\xi)\in\Omega\times\mathbf{R}} \frac{|f(x,\xi)|}{1+|\xi|^q} < +\infty.$$

For $0 < q \leqslant \frac{n+2}{n-2}$ and $f \in \mathcal{A}_q$, put

$$\Phi_f(u) = \int_\Omega \left(\int_0^{u(x)} f(x,\xi) d\xi \right) dx$$

and

$$J_f(u) = \frac{1}{2} \int_\Omega |\nabla u(x)|^2 dx - \Phi_f(u)$$

for all $u \in X$.

So, the functional J_f is continuously Gâteaux differentiable on X and one has

$$J'_f(u)(v) = \int_\Omega \nabla u(x)\nabla v(x)\mathrm{d}x - \int_\Omega f(x,u(x))v(x)dx$$

for all $u, v \in X$. Hence, the critical points of J_f in X are exactly the weak solutions of problem (P_f).

If $q < \frac{n+2}{n-2}$, the functional Φ_f is sequentially weakly continuous, by Rellich-Kondrachov theorem. However, Φ_f may be discontinuous with respect to the weak topology. In this connection, consider the following

EXAMPLE 1. If $f(x,\xi) = |\xi|^{q-1}\xi$ with $0 < q \leqslant \frac{n+2}{n-2}$, then Φ_f is discontinuous with respect to the weak topology.

In fact, if V is any neighbourhood of 0 in the weak topology of X, then V does contain an infinite-dimensional linear subspace F of X. Consequently, if we choose $u \in F \setminus \{0\}$, we have $\lambda u \in V$ for all $\lambda \in \mathbf{R}$ as well as

$$\lim_{\lambda \to +\infty} \Phi_f(\lambda u) = \lim_{\lambda \to +\infty} \frac{\int_\Omega |u(x)|^{q+1} dx}{q+1} \lambda^{q+1} = +\infty ,$$

and so Φ_f is weakly discontinuous at 0.

On the other hand, when f does not depend on ξ, the functional Φ_f is weakly continuous being linear and continuous. The above remarks then lead to the following natural question:

PROBLEM 1. Is there some $f \in \mathcal{A}_q$, with $q < \frac{n+2}{n-2}$, which is not of the form $f(x,\xi) = a(x)$, such that the functional Φ_f is continuous with respect to the weak topology of X ?

To formulate the next problem, denote by τ_s the topology on X whose members are the sequentially weakly open subsets of X. That is, a set $A \subseteq X$ belongs to τ_s if and only if for each $u \in A$ and each sequence $\{u_n\}$ in X weakly convergent to u, one has $u_n \in A$ for all n large enough.

PROBLEM 2. Is there some $f \in \mathcal{A}_q$, with $q < \frac{n+2}{n-2}$, such that, for each $\lambda > 0$ and $r \in \mathbf{R}$, the functional $J_{\lambda f}$ is unbounded below and the set $J_{\lambda f}^{-1}(r)$ has no isolated points with respect to the topology τ_s ?

The interest for the study of Problem 2 comes essentially from the following result:

THEOREM 1 ([6], Theorem 3). *Let $f \in \mathcal{A}_q$ with $q < \frac{n+2}{n-2}$. Then, there exists some $\lambda^* > 0$ such that the functional $J_{\lambda^* f}$ has local minimum with respect to the topology τ_s.*

If Φ_f is weakly continuous, then the conclusion of Theorem 1 becomes stronger: the topology τ_s can be replaced by the weak topology. This remark is a further motivation for the study of Problem 1.

In the light of Theorem 1, the relevance of Problem 2 is clear. Actually, if f was answering Problem 2 in the affirmative, then, by Theorem 1, for some $\lambda^* > 0$, the functional $J_{\lambda^* f}$ would have infinitely many local minima in the topology τ_s. Consequently, problem $(P_{\lambda^* f})$ would have infinitely many weak solutions.

It is also worth noticing that if $f \in \mathcal{A}_q$ with $q < \frac{n+2}{n-2}$ and $\lim_{\|u\| \to +\infty} J_f(u) = +\infty$, then the local minima of J_f in the strong and in the weak topology of X do coincide ([3], Theorem 1). On the other hand, if $f(x, \xi) = |\xi|^{q-1}\xi$ with $1 < q < \frac{n+2}{n-2}$, then, for some constant $\lambda > 0$, it turns out that 0 is a local minimum of $J_{\lambda f}$ in the strong topology but not in the weak one ([3], Example 2). However, I do not know any example of f for which J_f has a local minimum in the strong topology but not in τ_s.

To introduce the third problem (the most difficult, in my opinion), let me recall that in any vector space there is the strongest vector topology of the space ([1], p. 42).

PROBLEM 3. Denote by τ the strongest vector topology of X. Is there some $f \in \mathcal{A}_{\frac{n+2}{n-2}}$ such that the set

$$\{(u, v) \in X \times X : J_f'(u)(v) = 1\}$$

is disconnected in $(X, \tau) \times (X, \tau)$?

The motivation for the study of Problem 3 comes from the following result:

THEOREM 2 ([4], Theorem 1.2). *Let S be a topological space, Y a real topological vector space* (*with topological dual Y^**), *and $A: S \to Y^*$ a weakly-star continuous operator.*

Then, the following assertions are equivalent:

(i) *The set*

$$\{(s, y) \in S \times Y : A(s)(y) = 1\}$$

is disconnected.

(ii) *The set $S \setminus A^{-1}(0)$ is disconnected.*

Assume that $f \in \mathcal{A}_{\frac{n+2}{n-2}}$ have the property required in Problem 3. Since $J_f \in C^1(X)$, clearly the operator $J_f' : X \to X^*$ is τ-weakly-star continuous. Hence, by Theorem 2, the set $X \setminus (J_f')^{-1}(0)$ is τ-disconnected. Then, this implies, in particular, that the set $(J_f')^{-1}(0)$ is not τ-relatively compact ([5], Proposition 3), and hence is infinite. So, for such an f, problem (P_f) would have infinitely many weak solutions.

Of course, to recognize the disconnectedness of the set $\{(u,v)\in X\times X: J_f'(u)(v)=1\}$ in $(X,\tau)\times(X,\tau)$, it is enough to check that this set is disconnected in $(X,\tau_1)\times(X,\tau_1)$, where τ_1 is any vector topology on X (which, to be meaningful in view of Theorem 2, should also be stronger than the norm topology).

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